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On variational description of the trajectories of averaging quantum dynamical semigroups

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The family of averaging dynamical maps

The sequence of strongly continuous semigroups in Banach space X:

$$T_n(t),\ t \geqslant 0, \quad n \in \mathbf{N}$$

Let μ is nonnegative normalized finite additive measure on the measurable space $(\mathbf{N}, 2^{\mathbf{N}})$. We study the properties of the 1-parametric family of maps

$$T^{\mu}(t) = \int\limits_{\mathbf{N}} T_n(t) d\mu(n),\ t \geqslant 0,$$

where integration is defined in Pettis since.
The description of the set of limit points of the sequence of reqularizing semigroups.
The semigroup property and irreversibility property of the family of averaging dynamical maps $T^{\mu}(t),\ t \geqslant 0$.

The Cauchy problem for degenerated Schrodinger equation

The Cauchy problem for Schrodinger equation

$$i\frac{d}{dt}u(t) = \mathbf{L}u(t),\ t > 0, \tag{1}$$

$$u(+0) = u_0,\ u_0 \in H, \tag{2}$$

$\mathbf{L}$ is symmetric operator in the Hilbert space $H = L_2(R)$, and 2-nd order linear differential operator with nonnegative characteristic form.

The model problem (1), (2)

$$\mathbf{L}u(x) = \frac{\partial}{\partial x}(g(x)\frac{\partial}{\partial x}u + \frac{i}{2}a(x)u) + \frac{i}{2}a(x)\frac{\partial}{\partial x}u \qquad (3)$$

$$D(\mathbf{L}) = \{u \in W_2^1 : \ u|_{R_-} \in W_2^2(R_-)$$

$$(g(x)\frac{\partial u}{\partial x} + \frac{i}{2}a(x)u) \in W_2^1(R)\} \qquad (4).$$

Here $g(x) = \theta(-x)$, $a(x) = \alpha\theta(x)$, where $\alpha \in R$ and $\theta(x)$ is Heviside function.

L is densely defined closed symmetric operator with deficience indexes (n_-, n_+)

(1,0) if $\alpha < 0$; (0,0) if $\alpha = 0$; (0,1) if $\alpha > 0$.

The correctness of Cauchy problem

Theorem 1.

Let $\mathbf{L}$ is operator above. Then
1. $\alpha \leqslant 0$ (then $n_+ = 0$) $\Rightarrow e^{-it\mathbf{L}}$ is isometric semigroup. The problem (1), (2) $\exists$! solution $u(t) = e^{-it\mathbf{L}} u_0$.
2. $\alpha > 0$ (then $n_- = 0$) $\Rightarrow e^{it\mathbf{L}}$ is isometric semigroup, $e^{-it\mathbf{L}^*}$ is contractive semigroup, and

$$H = H_0 \oplus H_1,$$

where $H_0 = \overline{\bigcap_{t>0} (e^{it\mathbf{L}} H)}$, $H_1 = \overline{\bigcup_{t>0} \mathrm{Ker}(e^{-it\mathbf{L}^*})}$.
The solution of (1), (2) exists $\Leftrightarrow u_0 \in H_0 \Rightarrow$ unique.

Regularization (simple)

$$i\frac{d}{dt}u(t) = \mathbf{L}_\epsilon u(t),\ t > 0,\ \epsilon \in (0,1). \tag{5}$$

$$\mathbf{L}_\epsilon = \mathbf{L} + \epsilon\mathbf{\Delta}:\ \ g_\epsilon(x) = g(x) + \epsilon.$$

$$\mathbf{L}_\epsilon = \mathbf{L}_\epsilon^* \ \forall\ \epsilon \in (0,1). \quad \{u_\epsilon(t)\} = \{e^{-i\mathbf{L}_\epsilon t}u_0\}.$$

Theorem 2.

1. $\alpha \leqslant 0 \Rightarrow \forall\ T > 0,\ u_0 \in H \lim\limits_{\epsilon\to 0} \sup\limits_{t\in[0,T]} \|u_\epsilon(t) - u(t)\|_H = 0.$

2. $\alpha > 0 \Rightarrow \forall\ T > 0,\ u_0,\ v \in H$

$\lim\limits_{\epsilon\to 0} \sup\limits_{t\in[0,T]} |(v, u_\epsilon(t) - u^*(t))| = 0$, where $u^*(t) = e^{-i\mathbf{L}^* t}u_0$.

$\{u_\epsilon\}$ converges strongly $\Leftrightarrow u_0 \in H_0$.

If $u_0 \in H_1$ then $\lim\limits_{t\to+\infty} \|u^*(t)\|_H = 0.$

The set of quantum states

H is separable Hilbert space ($L_2(R)$).
$B(H)$ is Banach space of bounded linear operators in H.
$B^*(H)$ is Banach space conjugate to $B(H)$.

$\Sigma(H) = S_1(B^*(H)) \bigcap B^*_+(H)$ is the SET of quantum states.

$\Sigma_p(H) = \{\rho_u \in \Sigma(H),\ u \in S_1(H) :\ \rho(\mathbf{A}) = (u, \mathbf{A}u)_H\ \forall\ \mathbf{A} \in B(H)\}$ is the set of pure states.

$\Sigma_n(H) = \{\rho = \sum\limits_{k=1}^{\infty} p_k \mathbf{P}_{e_k},\ \{e_k\}\ is\ ONS\}$ is the set of normal states.

The non-normal states: KMS-states; Dixmer trace; The states after the measurement of observable with continuous spectrum.

Theorem S. $\rho \in \Sigma(H) \ \Rightarrow$

$\Rightarrow\ \exists\, \mu \in S_1(l^*_\infty) \bigcap (l^*_\infty)_+;\ \exists$ *the sequence* $\{e_k\}$ *of uniq vectors* :

$$\rho = \int\limits_{\mathbf{N}} \rho_{e_k} d\mu.$$

The weak* convergence of regularizing density operators

$u_\epsilon(t, u_0) = e^{-i\mathbf{L}_\epsilon t} u_0$ – the sequence of solutions of regularizing problems (2), (5).

$\rho_\epsilon(t, u_0) = \rho_{u_\epsilon(t,u_0)} \in \Sigma_p(H)$ – the sequence of regularizing density operators: $\langle \rho_\epsilon(t, u_0), \mathbf{A}\rangle = (u_\epsilon(t, u_0), \mathbf{A} u_\epsilon(t, u_0))\}$.

Weak* convergence: $\rho_\epsilon \hookrightarrow^* \rho \Leftrightarrow \langle \rho_\epsilon, \mathbf{A}\rangle \to \langle \rho, \mathbf{A}\rangle \ \forall \ \mathbf{A} \in B(H)$.

Theorem 3.

Let $\alpha > 0$. Then $\forall t > 0$, $\forall \{\epsilon_n\} \to 0 \ \exists \, u_0 \in S_1(H)$: $\{\rho_{\epsilon_n}(t, \rho_{u_0})\}$ diverges in weak topology.*

$$\forall t > 0, \ \forall \{\epsilon_n\} \to 0 \ \exists \, u_0 \in S_1(H) : \ \exists \, \mathbf{A} \in S_1(B(H)) :$$

$$\textit{the sequence } \{\langle \rho_{\epsilon_n}(t, \rho_{u_0}), \mathbf{A}\rangle\} \textit{ diverges}.$$

The sequence of regularizing quantum states as the random process

$E=(0,1)$; $(E,2^E)$ is measurable space.
Let $ba(E,2^E)$ be Banach space of finite additive measures on the measurable space $(E,2^E)$.

$$W(\mathbf{N})=(S_1(l_\infty^*/l_1))_+.$$

$$W(E)=\{\mu\in S_1(ba(E,2^E))\bigcap ba_+(E,2^E):$$
$$\mu(K)=0\,\forall\,K\subset E,\,0\notin\bar{K}\}.$$

We study the random process

$$\rho_\epsilon(t,\rho_{u_0}):\ (E,2^E,\mu)\to(B^*(H),\mathrm{Cyl}_{B(H)}(B^*(H))),\ t>0,\quad (6)$$

where $u_0\in S_1(H)$.

The structure of the set of limit points and the mean values of the process

The mean values of random process (6) is the Pettis integral

$$\rho^{\mu}(t, \rho_{u_0}) = \int_E \rho_\epsilon(t, \rho_{u_0}) d\mu(\epsilon), \; t > 0, \; \rho_{u_0} \in \Sigma_p(H). \qquad (7)$$

i.e. $\forall \; \mathbf{A} \in B(H): \; \langle \rho^{\mu}(t, \rho_{u_0}), \mathbf{A} \rangle = \int_E \langle \rho_\epsilon(t, \rho_{u_0}), \mathbf{A} \rangle d\mu(\epsilon)$.

We obtain the parametrization of the set of limit points by the measures from the set $W(E)$.

Theorem 4. *The state ρ is the limit point of the sequence $\{\rho_\epsilon(t, \rho_{u_0}), \; \epsilon \to 0\}$ iff ρ is the mean value (7) of the process (6) with some $\mu \in W(E)$.*

The family of mean dynamical maps $T^{\mu}(t),\ t > 0,\ \mu \in W(E)$

Let $t > 0$ and

$$T_{\epsilon}(t):\ \rho_{u_0} \to \rho_{\epsilon}(t, \rho_{u_0}); \qquad T^{\mu}(t):\ \rho_{u_0} \to \rho^{\mu}(t, \rho_{u_0})$$

are the sequence of regularizing dynamical semigroups and the family averaged dynamical maps of the space $B^{*}(H)$.

$$T^{\mu}(t) = \int_{E} T_{\epsilon}(t) d\mu,\ t > 0;\ \mu \in W(E)$$

For any $\mu \in W(E)$ and $t > 0$ the map $T^{\mu}(t)$ is continuous linear map of the space $B^{*}(H)$ with invariant set $\Sigma(H)$.

The mean trajectory of the averaged maps $T^{\mu}(t)$, $t \geqslant 0$, with the initial point $\rho_0 \in \Sigma(H)$ is the curve in Banach space $B^{*}(H)$: $\gamma^{\mu}_{\rho_0} = \{\rho^{\mu}(t, \rho_0) = T^{\mu}(t)\rho_0;\ t \geqslant 0\}$

The properties of the mean maps

Let $t > 0$, $H_0(t) = \mathrm{Im}(e^{i\mathbf{L}t})$, $H_1(t) = \mathrm{Ker}(e^{-i\mathbf{L}^*t})$.

Remark 5. Let the vector $u_0 \in S_1(H)$ has the nontrivial projections u_{0s} on the subspaces $H_s(t)$, $s = 0, 1$. Then for any measure $\mu \in W(E)$ the image of the circle

$$\mathcal{C}_{u_0} \equiv \{\rho_{0\alpha} = \rho_{u_{00}+e^{i\alpha}u_{01}},\, \alpha \in [0, 2\pi)\}$$

by the map $T^\mu(\tau)$ is one-point set as $\tau = t$ and is the circle for some $\tau > t$.

The maps $T^\mu(t)$, $t > 0$, don't possess the semigroup property and injectivity property.

The problem:
Can we obtain the trajectory $\rho^\mu(t, \rho_0)$ of the averaged map by the information about $\mu \in W(E)$ and $\rho^\mu(t_0, \rho_0)$ for some $t_0 > 0$?

Variational description of mean trajectories

Theorem 5. *There is the class of measures* $\mathcal{M} \subset W(E)$ *such that* $\forall\ \mu \in \mathcal{M}$ *the family of maps* $T^{\mu}(t), t > 0,$ *has the following properties.*
There is the functional $\Phi(t, \rho, r),\ (t, \rho, r) \in R \times \Sigma(H) \times \Sigma_p(H)$ *such that*
1. For any $\rho_0 \in \Sigma_p(H)$ *and any* $t > 0$ *the set of strong minimum points of the functional* $\Phi(t, T^{\mu}(t)\rho_0, \cdot)$ *is the curve* $\mathcal{C}(t) = \mathrm{argmax}(\Phi(t, T^{\mu}(t)\rho_0, \cdot)) \in \Sigma^p(H)$ *which is diffeomorphic to the circle.*
2. For any $\rho_0 \in \Sigma_p(H)$ *and any* $t > 0$ *there is the number* $t_1 \neq t,\ t_1 > 0$ *such that* $\rho_0 = \mathcal{C}(t) \bigcap \mathcal{C}(t_1)$.
3. The map $T^{\mu}(t)$ *is the isometric bijection of the set* $\Sigma_n(H_s(t))$ *onto the set* $T^{\mu}(t)(\Sigma_n(H_s(t)))$ *as* $s = 0, 1,$ *moreover*

$$T^{\mu}(t)[\mathrm{Extr}(\Sigma_n(H_s(t)))] = \mathrm{Extr}(T^{\mu}(t)\Sigma_n(H_s(t))).$$

$$T^{\mu}(t)\Sigma(H_0(t)) = \Sigma(H); \quad T^{\mu}(t)\Sigma_n(H_1(t)) \bigcap \Sigma_n(H) = \emptyset.$$

The nonlocal variational problem

Theorem 6. If the measure μ satisfy the condition $\mu \in M$ then the equality $\inf_{t>0} \sup_{r \in \Sigma_p(H)} \Phi(t, T^{\mu}(t)\rho_0, r) = 1$ is necessary and sufficient for inclusion $\rho_0 \in \Sigma_p(H)$.
If the condition $\rho_0 \in \Sigma_n(H)$ holds then there is the numbers $t_1, t_2 \in (0, +\infty)$ such that the equalities $\sup_{r \in \Sigma_p(H)} \Phi(t_i, \rho^{\mu}(t_i, \rho_0), r) = 1,\ i = 1, 2,$ is sufficient for the inclusion $\rho_0 \in \Sigma_p(H)$ and following equality holds

$$\rho_0 = \mathrm{argmax}(\Phi(t_1, T^{\mu}(t_1)\rho_0, \cdot)) \bigcap \mathrm{argmax}(\Phi(t_2, T^{\mu}(t_2)\rho_0, \cdot)).$$

Variational description of mean trajectories with the initial vector state in the subspace H_0

Let $t > 0$, $\mu \in W(E)$, $u_0 \in S_1(H)$. Can we find ρ_{u_0} by the information about the values

$$\langle \rho^{\mu}(t, \rho_{u_0}), \mathbf{A} \rangle, \ \mathbf{A} \in B(H)? \qquad (I)$$

Let $H_0(t) = \mathrm{Im}(e^{it\mathbf{L}})$. If $u_0 \in H_0(t)$ then we can find ρ_{u_0} by the information about the values (I).
Let $\{e_k\}$ is some ONB in the separable Hilbert space $H_0(t)$. Then we can find the ONS $\{f_k\}$ in the space H where $f_k = e^{-it\mathbf{L}^*}e_k$.
Since $u(t, u_0) = e^{-it\mathbf{L}^*}u_0$ is the solution of Cauchy problem (1), (2) then $\rho^{\mu}(t, \rho_{u_0}) = \rho_{u(t,u_0)}$ for any $\mu \in W(E)$. Therefore $\langle \rho^{\mu}(t, \rho_{u_0}), \mathbf{P}_{f_k} \rangle = |(e_k, u_0)|^2$ and we can find the absolute values of Fourier coefficients of the vector u_0. Hence for any $\epsilon > 0$ we can find the vector $v_{0\epsilon}$ such that $\|u_0 - v_{0\epsilon}\|_H < \epsilon$.

Variational description of mean trajectories with the initial vector state in the subspace H_1

Let $H_1(t) = (H_0(t))^{\perp} = \mathrm{Ker}(e^{-it\mathbf{L}^*})$. If $u_0 \in H_1(t)$ then we can find ρ_{u_0} by the information about the values (I).
Let $\{e_k\}$ is some ONB in the separable Hilbert space $H_1(t)$. Then $e^{-it\mathbf{L}_\varepsilon} e_k \to \theta_H \ \forall \ k \in \mathbf{N}$.
Lemma 1. For any $\sigma > 0$ there is the subsequence $\{\varepsilon_k\}$ such that the system of unit vectors $\{f_{1k}\}$ is σ-ONS: $\sum\limits_{k=1}^{\infty} \|f_{ik} - g_k\|^2 < \sigma^2$ for some ONS $\{g_k\}$. Here $f_{1k} = e^{-it\mathbf{L}_{\varepsilon_k}} e_1$.
Lemma 2. For any $\sigma > 0$ and any $m \in \mathbf{N}$ there is the subsequence $\{\varepsilon_k^{(m)}\}$ such that
1) the system of unit vectors $\{f_{jk}^{(m)}\}$ is 2^{-m}-ONS, where $f_{jk} = e^{-it\mathbf{L}_{\varepsilon_k^{(m)}}} e_j$, $j \in \{1, .., m\}$.
2) $\{\varepsilon_k^{(m+1)}\}$ is the subsequence of the sequence $\{\varepsilon_k^{(m)}\}$.

Variational description of mean trajectories with the initial vector state in the subspace H_1

Lemma 3. There is the measure $\mu \in W(E)$ such that
$\mu(\bigcup_{k\in\mathbf{N}} \varepsilon_k^{(m)}) = 1$ for any $m \in \mathbf{N}$.
Let $\mathcal{M} = \{\mu \in W(E) : \ \mu(\bigcup_{k\in\mathbf{N}} \varepsilon_k^{(m)}) = 1$ *for any* $m \in \mathbf{N}$.
Let $j \in \overline{1, n}$ for some nß$\mathbf{N}$ and $\mathbf{Q}^m(e_j) = \sum_{k=1}^{\infty} \mathbf{P}_{f_{jk}^{(m)}}$ for any $m \geqslant n$.
Then $\lim_{m\to\infty} \langle \rho^{\mu}(t, \rho_{u_0}), \mathbf{Q}^m(e_j)\rangle = |(e_j, u_0)|^2$ for any $\mu \in \mathcal{M}$.
Therefore we can find the absolute values of Fourier coefficients of the vector u_0. Hence for any $\epsilon > 0$ we can find the vector $v_{0\epsilon}$ such that $\|u_0 - v_{0\epsilon}\|_H < \epsilon$.

1. The influence of degeneration on the correctness of Cauchy problem.

2. Regularization. The behavior of the sequences of regularizing problem solutions and regularizing quantum states.

3. The limit points, set-valued maps and stochastic process.

4. The mean dynamical maps and the variational description of its trajectories.

THANK YOU!

The invariant means and finite additive measures

$(\mathbf{N}, 2^{\mathbf{N}})$ is measurable space.
$l_\infty^* = ba(\mathbf{N}, 2^{\mathbf{N}})$ is the Banach space conjugated to l_∞ (the space of finite additive measures).
$ca(\mathbf{N}, 2^{\mathbf{N}}) = l_1 \subset l_\infty^*$ is the space of countable additive measures.
$pba(\mathbf{N}, 2^{\mathbf{N}}) = l_\infty^* \backslash l_1 = \{\mu \in ba(\mathbf{N}, 2^{\mathbf{N}}) : \ \mu(K) = 0 \, \forall \, bounded \ K\}$ is the space of purely finite additive measures.
$V(\mathbf{N}) = S_1(l_\infty^*) \bigcap (l_\infty^*)_+$ is the set of nonnegative normalized measures.
$W(\mathbf{N}) = V(\mathbf{N}) \bigcap pba(\mathbf{N})$, $V_{ca}(\mathbf{N}) = V(\mathbf{N}) \bigcap ca(\mathbf{N})$.

The set of extreme points (E. Hewitt, K. Iosida, 1952)
$\mathrm{Extr}(V(\mathbf{N})) = V_0(\mathbf{N})$ –
$\mathrm{Extr}(W(\mathbf{N})) = W_0(\mathbf{N})$ is two-values measures (filters);
$\mathrm{Extr}(V_{ca}(\mathbf{N})) = V_{0,ca}(\mathbf{N})$ is atomic measures.

The invariant means and finite additive measures

Radon integral for scalar functions:
$a \in l_\infty,\ \mu \in W(\mathbf{N}),\ \Rightarrow \langle \mu, a \rangle = \int\limits_{\mathbf{N}} a_k d\mu.$
If the measure $\mu \in W(\mathbf{N})$ is invariant with respect to the group (of a shifts) then this integral is Banach Limit.
Let X is Banach space, X_* is it's pre-conjugate.
$\{u_k\}$ is the sequence: $\mathbf{N} \to X$.
The Pettis integral:

$$y = \int\limits_{\mathbf{N}} u_k d\mu \ \Leftrightarrow\ \langle y, h \rangle = \int\limits_{\mathbf{N}} \langle u_k, h \rangle d\mu \ \forall\ h \in X_*.$$

The subsets of a set $\Sigma(H)$ and the commutatativity of a states with an operator.

$$\Sigma_v(H) = \{\rho \in \Sigma(H) : \exists\, ONS\,\{e_k\}; \exists\, \mu \in V(\mathbf{N}) : \rho = \int_{\mathbf{N}} \rho_{e_k} d\mu(k)\}$$

$$\Sigma_n(H) = \{\rho \in \Sigma_v(H) : \mu \in V_{ca}(\mathbf{N})\}, \ -\mathrm{trace-class\,orerators\,in\,H};$$

$$\Sigma_{pba}(H) = \{\rho \in \Sigma_v(H) : \mu \in W(\mathbf{N})\} - \{\mathrm{is\,not\,operator\,from\,B(H)}\}.$$

Definition 1. $[\rho, \mathbf{A}] = 0 \ \Leftrightarrow \ \langle \rho, [A, B] \rangle = 0, \forall\, B \in B(H).$

$$\Sigma_{com}(H) = \{\rho \in \Sigma(H) : \exists\,\{e_k\} - ONB : [\rho, \mathbf{P}] = 0\,\forall\,\mathbf{P}\ \textit{with base}\{e_k\}\}.$$

On the structure of quantum state set: it's extreme points, orthogonal and generalize decomposition of a state.

Theorem 1. $\Sigma_v(H) = \Sigma_{com}(H) = \Sigma_n(H) \oplus \Sigma_{pba}(H)$

Theorem 2. (Extreme points of $\Sigma_{com}(H)$).

$$\rho_{\mu,\{e_k\}} \in \mathrm{Extr}(\Sigma_{com}(H)) \;\Leftrightarrow\; \mu \in \mathrm{Extr}(V(\mathbf{N})) = V_0(\mathbf{N}).$$

Theorem 3.
$\rho \in \Sigma(H) \;\Rightarrow\; \exists\, \mu \in V(\mathbf{N}) : \exists$ *the sequence* $\{e_k\}$ *such that*

$$\rho = \int\limits_{\mathbf{N}} \mathbf{P}_{e_k} d\mu.$$

$\rho \in \mathrm{Extr}(\Sigma(H)) \;\Rightarrow\; 1)\mu \in V_0(\mathbf{N})$; 2) $e_k \rightharpoonup 0$ by the filter $\mathcal{F}_\mu$.

The measurement of initial state by the observation of mean trajectory.

1 H_0) Let $\{e_k\}$ is ONB in the subspace $H_0(t)$, then
$e_{t,k} = e^{-it\mathbf{L}^*} e_k,\ k \in \mathbf{N}$, – ONB in H,
moreover, the equality $|(u_0, e_k)|^2 = \langle \rho^\mu(t, \rho_{u_0}), \mathbf{P}_{e_{t,k}} \rangle$ holds.

2 $H_1(t)$) Let $f_1 \in S_1(H_1(t))$, then $e^{-it\mathbf{L}_\epsilon} f_1 \rightharpoonup \theta_H$ as $\epsilon \to 0$.
$\Rightarrow\ \forall\ \sigma > 0\ \exists\ \{\epsilon_k\}:\ \epsilon_k \to 0;\ \{f_1(t,k) = e^{-it\mathbf{L}_{\epsilon_k}} f_1,\ k \in \mathbf{N}\}$ – σ-ON system.
Let $\{f_j\}$ is ONB in the subspace $H_1(t)$
$\Rightarrow\ \ \forall\ m \in \mathbf{N}\ \exists\ \{\epsilon_k^m\}:\ \epsilon_k^m \to 0;\ \{f_j(t,k,m) = e^{-it\mathbf{L}_{\epsilon_k^m}} f_j,\ k \in \mathbf{N},\ j \in \{1, 2, ..., m\}\}$ – 2^{-m}-ONS.

$$Q_m(f_j) = \sum_{k=1}^{\infty} \mathbf{P}_{f_j(t,k,m)}.$$

If the measure μ is concentrated on the set $E^m = \bigcup_{k=1}^{\infty} \epsilon_k^m$,
then $|\langle \rho^\mu(t, \rho_{u_0}), \mathbf{Q}_m(f_j) \rangle - |(u_0, f_j)|^2| < 2^{-m+1}$.

The test operators and functionals.

The sets E^m, $m \in \mathbf{N}$ is the base of the filter $\mathcal{F}$
Let $\{f_j\}$ is ONB in the space $H_1(t)$ and measure μ is concentrated on the set $E^m = \bigcup\limits_{k=1}^{\infty} \epsilon_k^m \ \forall \ m \in \mathbf{N}$.
$\Rightarrow \ \exists \ \{\epsilon_k\}: \ \epsilon_k \to 0; \ \ \forall \ m \in \mathbf{N}: \ \{f_j(t,k) = e^{-it\mathbf{L}_{\epsilon_k}} f_j, \ k \in \mathbf{N}, \ j \in \{1,2,...,m\}\}$ – 2^{-m}-ONS.
$Q_m(f_j) = \sum\limits_{k=1}^{\infty} \mathbf{P}_{f_j(t,k,m)}$.
If the measure μ is concentrated on the filter $\mathcal{F}$,
then $|\langle \rho^{\mu}(t, \rho_{u_0}), \mathbf{Q}_m(f_j)\rangle - |(u_0, f_j)|^2| < 2^{-m+1} \ \forall \ m \in \mathbf{N}$.