Measures on infinite-dimensional spaces and Bogoliubov equations

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H.Poincaré stressed that in statistical mechanics there is a fundamental contradiction: reversibility in assumptions and irreversibility in conclusions.

Differentiable measures.

Definition 1. A measure ν is called C-differentiable along a vector field h if there exists a measure ν_h' such that for every $u \in C$ the following formula of integration by parts holds:

$$\int u' h dv = -\int u d\nu_h'.$$

If $\nu'_h \ll \nu$ then the corresponding Radon-Nykodym derivative is called logarithmic derivative of ν along h and is denoted by β_h^ν.

Symplectic locally convex space (LCS):

(E, I), E − LCS , $I : E' \longrightarrow E$, $I^* = -I$.

Hamilton system $(E, I, \mathcal{H})$.

Hamilton equation:

$$f'(t) = I\mathcal{H}'f(t).$$

Liouville equation w.r.t. functions:

$$\frac{\partial F}{\partial t}(t) = \mathcal{L}_{\mathcal{H}}(F(t));$$

Liouville operator: $(\mathcal{L}_{\mathcal{H}}\Phi)(x) = \{\Phi, \mathcal{H}\}(x)$;

Poisson bracket: $\{\cdot, \cdot\}$,

$\{\Phi, \Psi\}(x) = \Phi'(x)(I(\Psi'(x)))$.

Liouville equations w.r.t. measures.

If G is a finite-dimensional factor space of E, then $\mathcal{A}_G(E)$ is the inverse image of the σ-algebra of the Borel subsets of G w.r.t. the canonical mapping $E \longrightarrow G$.

$\mathcal{A}(E) = \cup_{G \in \mathcal{G}} \mathcal{A}_G(E)$.

Cylindrical measure on E: $\nu : \mathcal{A}(E) \longrightarrow \mathbb{R}^+$,

if $G \in \mathcal{G}$ then the restriction of ν to $\mathcal{A}_G(E)$ is $\sigma-$additive.

The set of all cylindrical measures: $\mathcal{M}(E)$.

G-cylindrical functions and cylindrical functions.

A measure $\nu \in \mathcal{M}(E)$ is differentiable along $h \in E$, if there exists a function $\beta^\nu(h, \cdot)$ on E, which is called logarithmic derivative of ν along h, for which $\int_E f(x)\beta^\nu(h, x)\nu(dx) = -\int_E f'(x)h\nu(dx)$ for any cylindrical function that is differentiable along h and bounded together with $f'(\cdot)h$. If $k : E \to E$ is a vector field then

$$\beta_k^\nu(x) = \beta^\nu(k(x), x) + trk'(x)$$

. The measures $\beta^\nu(h, \cdot)\nu$ и $\beta_k^\nu(\cdot)\nu$ are called derivatives of ν along h and k and denoted by $\nu'h$ и $\nu'k$.

A Liouville theorem

Proposition 1. If a vector field k is Hamiltonian then

$$\beta_k^{\nu}(x) = \beta^{\nu}(k(x), x)$$

.

Proposition 2 (Liouville theorem). If F — is a canonical transformation, $\nu \in \mathcal{M}(E)$, νF is the image of ν w.r.t. F^{-1}, then

$$\frac{d(\nu F)}{d\nu}(x) = exp(\int_0^1 \beta^{\nu}(\mathcal{F}_1'(t,x), \mathcal{F}(t,x)dt),$$

where $\mathcal{F} : [0,1] \times E \to E$ is differentiable w.r.t. the first argument and $\mathcal{F}(0,x) = x$; $\mathcal{F}(1,x) = F(x)$

Liouville's equation for measures.

Definition. Liouville's equation for measures on the phase space E is the equation

$$\frac{\partial \nu}{\partial t}(t) = \mathcal{L}^*_{\mathcal{H}}(\nu(t))$$

w.r.t. functions of real variable taking values in a space of measures on E; here $\mathcal{L}^*_{\mathcal{H}}$ is the operator in a space of measures.

3. Equations w.r.t. finite-dimensional distributions.

Let there exist a symplectic basis $\mathcal{B} = \{e_k : k = 1, 2, ...\}$ in E; moreover let there exist a (pre)Hilbert structure and elements $\mathcal{B}$ form an o.n. basis. Finally we assume that span of every finite family of elements from $\mathcal{B}$ is equipped with the Lebesgue measure. For any finite family $\{e_{k_1}, ..., e_{k_n}\}$ of elements from $\mathcal{B}$, let $F_{k_1,...,k_n}$ be its span, $G_{k_1,...,k_n}$ be the symplectic factor space of $E \backslash F^c_{k_1,...,k_n}$ where $F^c_{k_1,...,k_n}$ is the closure of the span of elements of $\mathcal{B}$ not belonging to $\{e_{k_1}, ..., e_{k_n}\}$. Let $\mathcal{G}(\mathcal{B})$ be the collection of all such factor spaces.

The Liouville equation w.r.t measures on E is equivalent to the system of equations with respect to restrictions of the measures to subalgebras $\mathcal{A}_G(E),\ G \in \mathcal{G}(\mathcal{B})$.

But this system is equivalent to the Bogoliubov equations only if E is finite-dimensional.

For any $\mu \in \mathcal{M}(E)$ and for any $\{e_{k_1}, ..., e_{k_n}\} \subset \mathcal{B}$ let $f^{\mu}_{k_1,...,k_n}$ be the corresponding density of the probability on $G_{k_1,...,k_n}$.

Theorem 2. Let $\nu \in \mathcal{M}(E)$,

$\{e_{k_1}, ..., e_{k_n}\}$, $\{e_{r_1}, ..., e_{r_m}\} \subset \mathcal{B}$, let the Hamilton function $\mathcal{H}$ be $G_{r_1,...,r_m}$-cylindrical. Denote by $h_{r_1,...,r_m}$ the corresponding vector field on E, $h_{r_1,...,r_m} = I(\mathcal{H}'(\cdot))$. If $\{e_{r_1}, ..., e_{r_m}\} \subset \{e_{k_1}, ..., e_{k_n}\}$, then

$f^{\nu' h}_{k_1,...,k_n}(x) = (f^{\nu}_{k_1,...,k_n})'(x) h_{r_1,...,r_m}(x), \ x \in G_{k_1,...,k_n};$

if $\{e_{s_1}, ..., e_{s_p}\} = \{e_{r_1}, ..., e_{r_m}\} \backslash \{e_{k_1}, ..., e_{k_n}\} \neq \varnothing$, then

$$f^{\nu' h}_{k_1,...,k_n}(x) = \int (f^{\nu}_{\{r_1,...,r_m\} \cup \{k_1,...,k_n\}})'_{s_1,...,s_p}(...x_{s_1}, ..., x_{s_p}...) \cdot$$
$$h_{r_1,...,r_m}(...x_{s_1}, ..., x_{s_p}...) dx_{s_1}...dx_{s_p},$$

here $(\cdot)'_{s_1,...,s_p}$ is the derivative along $G_{s_1,...,s_p}$.

Bogoliubov type system of equations with respect to probability densities. Let $\mathcal{H} = \sum \mathcal{H}_{r_1,\ldots,r_n}$ where $\mathcal{H}_{r_1,\ldots,r_n}$ are $G_{r_1,\ldots,r_n}$-cylindrical function and let $h_{r_1,\ldots,r_m} = I(\mathcal{H}'_{r_1,\ldots,r_m}(\cdot))$.

Theorem 3. A $\mathcal{M}(E)$-valued function $\nu(\cdot)$ is a solution of the Liouville equation iff the functions $t \mapsto g_{r_1,\ldots,r_n}(t)$, defined by $g_{r_1,\ldots,r_n}(t) = f^{\nu(t)}_{r_1,\ldots,r_n}$ satisfies the following infinite system of equations:

$$g'_{k_1,\ldots,k_n}(t)(\cdot) = \sum \int (g_{\{r_1,\ldots,r_m\}\cup\{k_1,\ldots,k_n\}}(t))'_{s_1,\ldots,s_p}(\ldots x_{s_1},\ldots,x_{s_p}\ldots)\cdot$$

$$h_{r_1,\ldots,r_m}(\ldots x_{s_1},\ldots,x_{s_p}\ldots)dx_{s_1}\ldots dx_{s_p},$$

the sum is taken over all finite sets $\{r_1, ..., r_m\}$ of natural numbers and $\{s_1, ..., s_p\} = \{r_1, ..., r_m\} \setminus \{k_1, ..., k_n\}$; if $\{s_1, ..., s_p\} = \varnothing$ then we assume that the symbol of integration is absent.

Generalized Bogoliubov equations

Let the operator $\mathcal{L}^*_{\mathcal{H}}$ be defined on a space $\mathcal{M}_\infty(E)$ of infinite measures on E. The integrals are defined to be the limits of sets of integrals over subspaces $\{F_{r_1,...,r_n}\}$. If the measures on the subspaces $\{F_{r_1,...,r_n}\}$ are defined by the densities $\psi_{r_1,...,r_n}$ then we say that the family of functions $\{\psi_{r_1,...,r_n} : \{r_1,...,r_n\} \subset \mathbb{N}\}$ defines a measure ν on E. The family of functions is called adapted if for any $\{r_1,...,r_n\}$ and its proper subset $\{r_1,...,r_k\}$,
$\psi_{r_1,...,r_k}(x_1,...,x_k) = \lim \frac{1}{vol(V)} \int_V \psi_{r_1,...,r_n}(x_1,...,x_n) dx_{k+1}...dx_n$
if $vol(V) \to \infty$; here V is the centered ball in $F_{r_{k+1},...,r_n}$ and $vol(V)$ is its Lebesgue measure.

Theorem. For any $\{r_1, ..., r_n\}$ let $g_{r_1,...,r_n}$ be a function of real variable taking values in the space of bounded nonnegative continuous functions on $F_{r_1,...,r_n}$ and let for any t the family $g_{r_1,...,r_n}(t)$ is adapted and define the unique measure $\nu(t)$ on E. Then $\nu(\cdot)$ is a solution of the Liouville equation for measures if the family $g_{r_1,...,r_n}$ is the solution of the following system of equations:

$$g'_{k_1,...,k_n}(t)(\cdot) = \sum_{m=1}^{\infty}(\sum_{\{r_1,...,r_m\} \subset\{k_1,...,k_n\}}(g_{k_1,...,k_n}(t))'(\cdot)h_{r_1,...,r_m}(\cdot)+ \sum_{\{j_1,...,j_p\}=\{r_1,...,r_m\}\cap\{k_1,...,k_n\}} \lim_{N\to\infty} \frac{1}{N^{m-p}} \sum_{\{r_1,...,r_m\}\backslash\{k_1,...,k_n\}\cap\{1,...,N\}} \int(g_{\{r_1,...,r_m\}\cup\{k_1,...,k_n\}}(t))'_{s_1,...,s_{m-p}}(...x_{s_1}, ..., x_{s_{m-p}}...) \cdot h_{r_1,...,r_m}(...x_{s_1}, ..., x_{s_{m-p}}...)dx_{s_1}...dx_{s_{m-p}}.$$

Definition 1. The Wigner function W_T generated by a density operator (=state) T is defined by

$$(T\varphi)(q) = \int_P \int_Q W_T(\frac{q_1 + q}{2}, p) e^{-ip(q_1 - q)} \varphi(q_1) dq_1 dp$$
$$= \int_Q \rho_T(q, q_1) \varphi(q_1) dq_1. \quad (1)$$

Definition 2. The Wigner function W_T generated by a density operator T given by an integral kernel ρ_T is defined by

$$W_T(q,p) = \frac{1}{(2\pi)^n} \int_Q \rho_T(q - \frac{1}{2}r, q + \frac{1}{2}r) e^{irp} dr.$$

As T is selfadjoint its kernel ρ is Hermitian and hence the range of W_T is in $\mathbb{R}$.

Definition 3. The Wigner function is the integral kernel of the linear functional $F \mapsto trT\hat{F}$ on the vector space of Weyl's symbols of (bounded) pseudodifferential operators in $L_2(Q)$.

$$trT\hat{F} = \int_P \int_Q W_T(q,p)F(q,p)dqdp. \tag{2}$$

Definition 4. The Weyl operator $\mathcal{W}(h)$ in $L_2(Q)$, generated by $h \in E$, is defined by: $\mathcal{W}(h) = e^{-i\hat{h}}$. The Weyl function $\hat{\mathcal{W}}$ generated by a state T is the function on E, defined by

$$\hat{\mathcal{W}}_T(h) = trT\mathcal{W}(h).$$

The Wigner function W_T is the inverse Fourier transform of the Weyl function:

$$W_T(q,p) = \frac{1}{(2\pi)^n} \int_P \int_Q e^{i(q\bar{p}+p\bar{q})} \mathcal{W}_T(\bar{q},\bar{p}) d\bar{q} d\bar{p}.$$

For quantum systems with quadratic Hamiltonians the evolution of the Wigner functions coincides with the evolution of the density of a probability distribution on the phase space of the corresponding classical Hamiltonian system.

Wigner measures.

Let $E = Q \times P$, where the LCS Q and P are such that $P = Q^*$ and $Q = P^*$; hence $E^* = P \times Q$ and the mapping $J : E \to E^*, (q, p) \mapsto (p, q)$ is an isomorphism. Let also the mapping $I : E^* \to E$ be defined by $I(p, q) = (q, -p)$. The LCS Q (resp. P) is called the configuration space (resp., the momentum space) of the Hamiltonian system $(E, I, \mathcal{H})$. If $q_1, q_2 \in Q, p_1, p_2 \in P$ then the value that the linear functional $(p_1, q_1) = J(q_1, p_1)$ takes at the element (q_2, p_2) is denoted by by $p_1 q_2 + q_1 p_2$. We assume that the Hilbert space H of the corresponding quantum system is the complex space $L_2(Q, \mu)$ where μ is a P-cylindrical measure on Q; in order to define infinite-dimensional pseudodifferential operators we assume that this is a Gaussian measure and use the ideology of the Hida White noise analysis calculus. Nevertheless this measure does not appear in the final formulae. Let the symbol T denote the von Neumann density operator (=trace class positive operator in H whose trace is equal to one) that defines a state of the system, and let ρ_T denote the integral kernel of the density operator.

If η is a (X^*)-cylindrical measure on a LCS X and $D(\eta)$ is the collection of all vectors along which the measure η is differentiable then the generalized density of η is a scalar function F_η on $D(\eta)$ whose logarithmic derivative along any $h \in D(\eta)$ is equal to $\beta^\eta(h, \cdot)$. Even for Gaussian measures the generalized density is defined only up to a multiplicative constant. One can show that if η is the Gaussian measure whose Fourier transform $\tilde{\eta}$ is defined by $\tilde{\eta}(z) = \exp\left(-\frac{1}{2}\langle zB(z)\rangle\right)$, where B is a linear mapping of X^* into X, then $F_\eta(x) = C\exp\left(-\frac{1}{2}\langle xB^{-1}(x)\rangle\right)$. This result shows that the Gaussian measure can be defined by its generalized density. Below we use the generalized density of the Gaussian measure in order to define pseudodifferential operators in $L_2(Q,\mu)$.

Let the measure μ be the Gaussian measure defined by its generalized density as follows: $F_\mu(q) = \exp(-\frac{1}{2}\langle qB^{-1}(q)\rangle)$ where $B \in L(P,Q)$ and let ν be a Q-cylindrical Gaussian measure on P defined by $F_\nu(p) = \exp(-\frac{1}{2}\langle q(B^*)^{-1}(q)\rangle)$. It is well known that if Q and P are Hilbert spaces then μ and

ν are σ-additive if and only if B is a (positive) trace class operator.

For each "good enough" scalar function (we will not formulate the corresponding analytical assumptions) $\mathcal{H}$ on $E(= Q \times P)$ the symbol $\hat{F}$ denotes the pseudodifferential operator in $L_2(Q, \mu)$ (which is supposed to be essentially selfadjoint), whose Weyl symbol is $\mathcal{H}$. This means that if $\varphi \in dom\hat{\mathcal{H}}(\subset L_2(Q, \mu))$, then

$$(\hat{\mathcal{H}}\varphi)(q) = \int_P \int_Q \mathcal{H}(\frac{q_1 + q}{2}, p) e^{-ip(q_1 - q)} \varphi(q_1)(F_\mu(q))^{-\frac{1}{2}}$$

$$(F_\mu(q_1))^{-\frac{1}{2}}(F_\nu(p))^{-1}\mu(dq_1)\nu(dp).$$

The integral ar r.h.s. is defined as follows:

$\int_P \int_Q \mathcal{H}(\frac{q_1+q}{2}, p) e^{-ip(q_1-q)} \varphi(q_1)(F_\mu(q))^{-\frac{1}{2}}$
$(F_\mu(q_1))^{-\frac{1}{2}}(F_\nu(p))^{-1}\mu(dq_1)\nu(dp)$
$= \lim_{n\to\infty} c_n \int_{P_n} \int_{Q_n} \mathcal{H}(\frac{q_1+q}{2}, p) e^{-ip(q_1-q)} \varphi(q_1)(F_\mu(q))^{-\frac{1}{2}}$
$(F_\mu(q_1))^{-\frac{1}{2}}(F_\nu(p))^{-1}\mu(dq_1)\nu(dp)$, where

$$c_n^{-1} = \int_{P_n} \int_{Q_n} e^{-ip(q_1-q)}(F_\mu(q))^{-\frac{1}{2}}(F_\mu(q_1))^{-\frac{1}{2}}(F_\nu(p))^{-1}\mu(dq_1)\nu(dp)$$

and $Q_n \times P_n = F_{k_1,\ldots,k_n}$; we assume that for any n the subspace $F_{k_1,\ldots,k_n}$ is contained in the domain of the integrands of the latter finite-dimensional integrals and use the regularization, of finite=dimensional integrals, which are defined by the following way.

If $f \in L_1^{loc}(\mathbb{R}^n)$, then we say that the integral $\int_{\mathbb{R}^n} f(x)dx$

exists if for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, for which $\varphi(0) = 1$, the limit $\lim_{a\to\infty} \int_{\mathbb{R}^n} \varphi(ax)f(x)dx$ exists and then by definition $\int_{\mathbb{R}^n} f(x)dx = \lim_{a\to\infty} \int_{\mathbb{R}^n} \varphi(ax)f(x)dx$ (the definition does not depend on the choice of φ).

For any $h \in E$ the symbol $\hat{h}$ denotes the pseudodifferential operator in $L_2(Q)$, whose Weyl symbol is $Jh(\in E^*)$; in particular if $h = q + p(= (q,p))$ then $\hat{h} = \hat{p} + \hat{q}$. Let us mention that if $q_0 \in Q$ then $\hat{q}_0$ is the operator of *the momentum in the direction of* q_0 (but not of the coordinate).

Remark 9 Let $\bar{E} = \{\hat{h} : h \in E\}$; then the mapping $F^- : Jh \mapsto \hat{h}, E^* \to \bar{E}$ is a liner isomorphism (we assume that $\bar{E}$ is equipped with the natural structure of a vector space). The extension of F^{-} to a linear mapping, of the space generated by E^{*} and the function on E whose values at each points are equal to one, into the space of operators in $L_2(Q, \mu)$, defined by the assumption that the image of this function is the multiplication by i, is called the Scrödinger representation of the canonical commutation relations.

Definition 3. The Weyl operator $\mathcal{W}(h)$ generated by $h \in E$ is defined by: $\mathcal{W}(h) = e^{-i\hat{h}}$. The Weyl function $\mathcal{W}_T$ corresponding to the state (=density operator) T of the quantum system, whose Hilbert space is $L_2(Q, \mu)$, is the function on E, defined by

$$\mathcal{W}_T(h) = trT\mathcal{W}(h).$$

Definition 4. The Wigner measure W_T generated by the state (=density operator) T of the quantum system, whose Hilbert space is $L_2(Q, \mu)$, is the image, with respect to the mapping $J^{-1} : E^* \to E$, of the E-cylindrical measure on E^* whose Fourier transform is the Weyl function. This means that

$$\int_{Q\times P} e^{i(p_1 q_2 + q_1 p_2)} W_T(dq_1 dp_1) = \mathcal{W}_T(q_2, p_2).$$

Remark 10. One can show that the Wigner measure can also be defined as the integral kernel of the linear functional

$F \mapsto trT\hat{F}$ on the vector space of the Weyl symbols of (bounded) pseudodifferential operators in $L_2(Q,\mu)$. This means that for any such Weyl's symbol F the following identity holds

$$trT\hat{F} = \int_P \int_Q F(q,p) W_T(dqdp) \tag{3}$$

(in finite-dimensional spaces the Wigner's measure can be substituted by its density, which is called the Wigner function).

Remark 11. The measure W_T^Q on Q defined by $W_T^Q(dq) = \int_P W_T(dqdp)$ is the (cylindrical) probability on Q describing the distribution of results of measurements of the coordinates.

To formulate the equation describing the evolution of the Wigner measure we need a definition of what one could call a function of the Poisson bracket. Below we use some topological tensor powers of the phase space but we do not discuss the topologies of them. We use the assumptions and definitions of sections 3 and 4.

Let, for any $n \in \mathbb{N}$, the symbol $I^{\otimes n}$ denotes the mapping, of a proper subspace of $\mathcal{B}_n(E)$, into $E^{\otimes n}$ generated by n-th tensorial power of I (here $E^{\otimes n}$ is a topological tensor product of n copies of E). Let moreover, for any two scalar functions F and G on E,

$$\{F, G\}^{(n)}(x) = F^{(n)}(x) I^{\otimes n} G^{(n)}(x)$$

and

$$\mathcal{L}_G^{(n)} H(x) = \{F, G\}^{(n)}(x)$$

(of course, $\mathcal{L}_G^n \neq \mathcal{L}_G^{(n)}$). Finally let, for any $a > 0$, the operator $(sin)a\mathcal{L}_G$ be defined by

$$(sin)a\mathcal{L}_G = \sum_{n=1}^{\infty} \frac{a^{2n-1}}{(2n-1)!} \mathcal{L}_G^{(2n-1)}$$

(we do not discuss now in which sense the series converges) and let $(sin)a\mathcal{L}_G^*$ be the operator in a space of E^*-cylindrical measures on E, which is adjoint to $(sin)a\mathcal{L}_G$.

Theorem 5. The dynamics of the Wigner measure is governed by the following equation:

$$W_T'(t) = 2(sin)\frac{1}{2}\mathcal{L}_{\mathcal{H}}^*(W_T).$$

The proof can be obtained by combination of technique of the theory of differentiable measures and some methods of developing equations describing the evolution of the Wigner function [4], [5], [6].

Theorem 6. Let $\nu \in \mathcal{M}_{\mathcal{G}}(E)$, $\{e_{k_1}, ..., e_{k_n}\}, \{e_{r_1}, ..., e_{r_m}\} \subset \mathcal{B}$ and let the Hamiltonian function $\mathcal{H}$ be $G_{r_1,...,r_m}$-cylindrical. If $\{e_{r_1}, ..., e_{r_m}\} \subset \{e_{k_1}, ..., e_{k_n}\}$, then

$$f_{k_1,...,k_n}^{2(sin)\frac{1}{2}\mathcal{L}_{\mathcal{H}}^*(\nu)}(x) = 2(sin)\frac{1}{2}\mathcal{L}_{\mathcal{H}}(f_{k_1,...,k_n}^{\nu})(x), \ x \in G_{k_1,...,k_n},$$

and if $\{e_{s_1}, ..., e_{s_p}\} = \{e_{r_1}, ..., e_{r_m}\} \backslash \{e_{k_1}, ..., e_{k_n}\} \neq \varnothing$,

then $f_{k_1,...,k_n}^{2(sin)\frac{1}{2}\mathcal{L}^*_{\mathcal{H}}(\nu)}(x) =$
$\int 2(sin)\frac{1}{2}\mathcal{L}_{\mathcal{H}_{r_1,...,r_m}}(f^{\nu}_{\{r_1,...,r_m\}\cup\{k_1,...,k_n\}})(...x_{s_1},...,x_{s_p}...)dx_{s_1}...dx_{s_p}$.

Theorem 7. A function $W(\cdot)$ taking values in $\mathcal{M}_{\mathcal{G}}(E)$ is a solution of the equation governing the dynamics of the Wigner measure if and only if the functions $t \mapsto g_{r_1,...,r_n}(t)$, defined by the equalities $g_{r_1,...,r_n}(t) = f_{r_1,...,r_n}^{2(sin)\frac{1}{2}\mathcal{L}^*_{\mathcal{H}}(W(t))}$ satisfy the following infinite system of differential equations:

$$g'_{k_1,...,k_n}(t)(\cdot) =$$

$$\sum\int 2(sin)\frac{1}{2}\mathcal{L}_{\mathcal{H}_{r_1,...,r_m}}(g_{\{r_1,...,r_m\}\cup\{k_1,...,k_n\}}(t))(...x_{s_1},...,x_{s_p}...)dx_{s_1}...dx_{s_p},$$

where the summation is over all finite sets $\{r_1,...,r_m\}$ of natural numbers for which $\{r_1,...,r_m\}\cap\{k_1,...,k_n\} \neq \varnothing$, and, in accordance with what has been said above, if $\{r_1, ..., r_m\} \cap \{k_1, ..., k_n\} = \{r_1, ..., r_m\}$, then the integral sign is assumed to be missing.

This follows from the theorems 5 and 6.

Suppose that the operator $(sin)\mathcal{L}^*_{\mathcal{H}}$ is defined on $\mathcal{M}^\infty(E)$.

Theorem 8. Let, for every finite set $\{r_1, ..., r_n\}$ of natural numbers, $g_{r_1,...,r_n}$ be a function of a real argument taking values in the space of (bounded continuous) functions on $F_{r_1,...,r_n}$, and, for each t the family of functions $g_{r_1,...,r_n}(t)$ is compatible and determines a unique measure $w(t) \in \mathcal{M}^\infty(E)$. Then, the function $w(\cdot)$ is a solution of the equation governing the dynamics of the Wigner measure, if the family of functions $g_{r_1,...,r_n}$ is a solution of the system of equations

$$g'_{k_1,...,k_n}(t)(\cdot) = \sum_{m=1}^{\infty}(\sum_{\{r_1,...,r_m\} \subset\{k_1,...,k_n\}} 2(sin)\frac{1}{2}\mathcal{L}_{\mathcal{H}_{r_1,...,r_m}}(g_{k_1,...,k_n}(t))(\cdot)+$$

$$+ \sum_{\{j_1,...,j_p\}=\{r_1,...,r_m\}\cap\{k_1,...,k_n\}} \lim_{N\to\infty} \frac{1}{N^{m-p}} \sum_{\{r_1,...,r_m\}\setminus\{k_1,...,k_n\}\cap\{1,...,N\}}$$

$$\int 2(sin)\frac{1}{2}\mathcal{L}_{\mathcal{H}_{r_1,...,r_m}}(g_{\{r_1,...,r_m\}\cup\{k_1,...,k_n\}}(t))(...x_{s_1}, ..., x_{s_{m-p}}...)dx_{s_1}...dx_{s_{m-p}}.$$

If it is not assumed that the Plank constant is equal to one then it is necessary to substitute "$2(\sin)\frac{1}{2}$" by "$\frac{2}{\hbar}(\sin)\frac{\hbar}{2}$", where $\hbar$ is the Plank constant.

Remark 12. From Theorem 8, which is similar to theorem 4, one can deduce a quantum version of the classical Bogolyubov system of equations and also some other similar systems of equations. It is also worth noticing that the integrals $\int_P w(t)(dq, dp)$ are not probabilities and hence (Remark 11) the measures $w(t)$ are not Wigner measures.

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Спасибо

Thank you