Quantum Dissipation from Power-Law Memory

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International Conference
"Irreversibility Problem in Classical and Quantum Dynamical Systems"

(Moscow, December 8-10, 2011)

A new quantum dissipation model based on memory mechanism is suggested. Dynamics of open and closed quantum systems with power-law memory is considered. An example of quantum oscillator with linear friction and power-law memory is considered.

The processes with power-law memory are described by using integration and differentiation of non-integer orders, by methods of fractional calculus. Fractional calculus is a theory of integrals and derivatives of any arbitrary real (or complex) order. It has a long history from 30 September 1695, when the derivatives of order $\alpha = 1/2$ has been described by Leibniz in a letter to L'Hospital The fractional differentiation and fractional integration go back to many great mathematicians such as Leibniz, Liouville, Riemann, Abel, Riesz, Weyl.

B. Ross, *A brief history and exposition of the fundamental theory of fractional calculus*, Lecture Notes in Mathematics. Vol.457. (1975) 1-36.

J.T. Machado, V. Kiryakova, F. Mainardi, *Recent History of Fractional Calculus*, Communications in Nonlinear Science and Numerical Simulations. Vol.16. (2011) 1140-1153.

Special Journals

- **"Fractional Calculus and Applied Analysis";**
- **"Fractional Differential Calculus";**
- **"Communications in Fractional Calculus".**

Mathematics Books

- **S.G. Samko, A.A. Kilbas, O.I. Marichev,** *Integrals and Derivatives of Fractional Order and Applications* **(Nauka i Tehnika, Minsk, 1987); and** *Fractional Integrals and Derivatives Theory and Applications* **(Gordon and Breach, New York, 1993)**
- **A.A. Kilbas, H.M. Srivastava, J.J. Trujillo,** *Theory and Applications of Fractional Differential Equations* **(Elsevier, Amsterdam, 2006).**

Physics Books

Applications of Fractional Calculus in Physics are described in the books:

- **R. Hilfer (Ed.),** *Applications of Fractional Calculus in Physics* **(World Scientific, Singapore, 2000).**
- **V.V. Uchaikin,** *Method of Fractional Derivatives* **(Artishok, Ulyanovsk, 2008) in Russian.**
- **A.C.J. Luo, V.S. Afraimovich (Eds.),** *Long-range Interaction, Stochasticity and Fractional Dynamics* **(Springer, Berlin, 2010)**
- **F. Mainardi,** *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models* **(World Scientific, Singapore, 2010).**
- **V.E. Tarasov,** *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media* **(Springer, New York, 2011).**
- **J. Klafter, S.C. Lim, R. Metzler (Eds.),** *Fractional Dynamics. Recent Advances* **(World Scientific, Singapore, 2011).**
- **V.E. Tarasov,** *Theoretical Physics Models with Integro-Differentiation of Fractional Order* **(IKI, RCD, 2011) in Russian**

In general, many usual properties of the ordinary (first-order) derivative D_t are not realized for fractional derivative operators D_t^α. For example, a product rule, chain rule, semigroup property have strongly complicated analogs for the operators D_t^α.

There are many different definitions of fractional integrals and derivatives of non-integer orders.

A generalization of Cauchy's differentiation formula

Let G be an open subset of the complex plane $\mathbb{C}$, and $f : G \to \mathbb{C}$ is a holomorphic function:

$$f^{(n)}(x) = \frac{n!}{2\pi i} \oint_L \frac{f(z)}{(z-x)^{n+1}}\, dz. \tag{1}$$

A generalization of (1) has been suggested by Sonin (1872) and Letnikov (1872) in the form

$$D_x^\alpha f(x) = \frac{\Gamma(\alpha+1)}{2\pi i} \oint_L \frac{f(z)}{(z-x)^{\alpha+1}}\, dz, \tag{2}$$

where $\alpha \in \mathbb{R}$ and $\alpha \neq -1, -2, -3, ...$ See Theorem 22.1 in the book by Samko, Kilbas, and Marichev. Expression (2) is also called Nishimoto derivative.

A generalization of finite difference

The differentiation of integer order n can be defined by

$$D^n_x f(x) = \lim_{h\to 0} \frac{\Delta^n_h f(x)}{h^n}, \quad \Delta^n_h f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-kh). \tag{3}$$

The difference of a fractional order $\alpha > 0$ is defined by the infinite series

$$\Delta^\alpha_h f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x-kh), \quad \binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)}. \tag{4}$$

The left- and right-sided Grünwald-Letnikov (1867,1868) derivatives of order $\alpha > 0$ are defined by

$${}^{GL}D^\alpha_{x\pm} f(x) = \lim_{h\to 0} \frac{\nabla^\alpha_{\mp h} f(x)}{h^\alpha}. \tag{5}$$

If

$$|f(x)| < c(1+|x|)^{-\mu}, \quad \mu > |\alpha|.$$

then the series (4) can be used for $\alpha < 0$ and Eq. (5) defines Grünwald-Letnikov fractional integral. If $f(x) \in L_p(\mathbb{R})$, where $1 < p < 1/\alpha$ and $0 < \alpha < 1$, then (5) can be represented by

$${}^{GL}D^\alpha_{x\pm} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x\mp z)}{z^{\alpha+1}} dz.$$

A generalization by Fourier transform

If we define the Fourier transform operator $\mathcal{F}$ by

$$(\mathcal{F}f)(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt, \tag{6}$$

then the Fourier transform of derivative of integer order n is

$$(\mathcal{F}D_x^n f)(\omega) = (i\omega)^n(\mathcal{F}f)(\omega).$$

Therefore

$$D_x^n f(x) = \mathcal{F}^{-1}\{(i\omega)^n(\mathcal{F}f)(\omega)\}.$$

For $f(t) \in L_1(\mathbb{R})$, the left- and right-sided Liouville fractional integrals and derivatives can be defined by the relations

$$(I_\pm^\alpha f)(x) = \mathcal{F}^{-1}\Big(\frac{1}{(\pm i\omega)^\alpha}(\mathcal{F}f)(\omega)\Big), \tag{7}$$

$$(D_\pm^\alpha f)(x) = \mathcal{F}^{-1}\Big((\pm i\omega)^\alpha(\mathcal{F}f)(\omega)\Big), \tag{8}$$

where $0 < \alpha < 1$ and

$$(\pm i\omega)^\alpha = |\omega|^\alpha \exp\Big(\pm sgn(\omega)\frac{i\,\alpha\,\pi}{2}\Big).$$

The Liouville fractional integrals (7) can be represented by

$$(I^{\alpha}_{\pm}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} z^{\alpha-1} f(x \mp z) dz. \tag{9}$$

The Liouville fractional derivatives (8) are

$$(D^{\alpha}_{\pm}f)(x) = D^n_x (I^{n-\alpha}_{\pm} f)(x).$$

Therefore

$$(D^{\alpha}_{\pm}f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^{\infty} z^{n-\alpha-1} f(x \mp z) dz, \tag{10}$$

where $n = [\alpha] + 1$**.**

Caputo derivative

We can define the derivative of fractional order α by

$$ {}^{C}D_{\pm}^{\alpha} f(t) = I_{\pm}^{n-\alpha}(D_t^n f)(t). $$

For $x \in [a, b]$ the left-sided Caputo fractional derivative of order $\alpha > 0$ is defined by

$$ {}_a^C D_t^{\alpha} f(t) = {}_a I_t^{n-\alpha} D_t^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{d\tau\, D_\tau^n f(\tau)}{(t-\tau)^{\alpha-n+1}}, \tag{11} $$

where $n-1 < \alpha < n$, and ${}_a I_t^{\alpha}$ is the left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ that is defined by

$$ {}_a I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau) d\tau}{(t-\tau)^{1-\alpha}}, \quad (a < t). $$

The Riemann-Liouville fractional derivative has some notable disadvantages in applications such as nonzero of the fractional derivative of constants,

$$ {}_0D_t^{\alpha} C = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} C, $$

which means that dissipation does not vanish for a system in equilibrium. The Caputo fractional differentiation of a constant results in zero

$$ {}_0^C D_t^{\alpha} C = 0. $$

The desire to use the usual initial value problems

$$ f(t_0) = C_0, \quad (D_t^1 f)(t_0) = C_1, \quad (D_t^2 f)(t_0) = C_2, ... $$

lead to the application Caputo fractional derivatives instead of the Riemann-Liouville derivative.

A physical interpretation of equations with derivatives and integrals of non-integer order with respect to time is connected with the memory effects.

Let us consider the evolution of a dynamical system in which some quantity $A(t)$ is related to another quantity $B(t)$ through a memory function $M(t)$ by

$$A(t) = \int_0^t M(t-\tau)B(\tau)d\tau. \tag{12}$$

This operation is a particular case of composition products suggested by Vito Volterra. In mathematics, equation (12) means that the value $A(t)$ is related with $B(t)$ by the convolution operation

$$A(t) = M(t) * B(t).$$

Equation (12) is a typical equation obtained for the systems coupled to an environment, where environmental degrees of freedom being averaged.

Let us consider the limiting cases widely used in physics.

(1) The absence of the memory: For a system without memory, the time dependence of the memory function is

$$M(t-\tau) = M(t)\,\delta(t-\tau), \tag{13}$$

where $\delta(t-\tau)$ is the Dirac delta-function. The absence of the memory means that the function $A(t)$ is defined by $B(t)$ at the only instant t. In this case, the system loses all its values of quantity except for one. Using (12) and (13), we have

$$A(t) = \int_0^t M(t)\delta(t-\tau)B(\tau)d\tau = M(t)B(t). \tag{14}$$

Expression (14) corresponds to the well-known physical process with complete absence of memory. This process relates all subsequent values to previous values through the single current value at each time t.

(2) Complete memory: If memory effects are introduced into the system, then the delta-function turns into some function with the time interval during which $B(t)$ affects on the function $A(t)$. Let $M(t)$ be the step function

$$M(t-\tau) = t^{-1}[\theta(\tau) - \theta(t-\tau)], \tag{15}$$

where $\theta(t)$ is the Heaviside function, also called the unit step function. The Heaviside function $\theta(t)$ is a discontinuous function whose value is zero for negative argument and one for positive argument. In equation (15), the factor t^{-1} is chosen to get normalization of the memory function to unity:

$$\int_0^t M(\tau)d\tau = 1.$$

Then in the evolution process the system passes through all states continuously without any loss. In this case,

$$A(t) = \frac{1}{t}\int_0^t B(\tau)d\tau,$$

and this corresponds to a complete memory.

(3) Power-law memory: The power-like memory function is defined by

$$M(t-\tau) = M_0\,(t-\tau)^{\varepsilon-1}, \tag{16}$$

where M_0 is a real parameter. Substitution of (16) into (12) gives the temporal fractional integral of order ε:

$$A(t) = \lambda I_t^{\varepsilon} B(t) = \frac{\lambda}{\Gamma(\varepsilon)} \int_0^t (t-\tau)^{\varepsilon-1} B(\tau) d\tau, \quad 0 < \varepsilon < 1, \tag{17}$$

where $\lambda = \Gamma(\varepsilon) M_0$. The memory determines an interval $[0,t]$ during which $B(\tau)$ affects $A(t)$.

Equation (17) is a special case of relation for $A(t)$ and $B(t)$, where $A(t)$ is directly proportional to $M(t) * B(t)$. In a more general case, the values $A(t)$ and $B(t)$ can be related by the equation

$$f(A(t), M(t) * D_t^n B(t)) = 0, \tag{18}$$

where f is a smooth function. In this case equation (18) gives the relation $f(A(t),\ {}_0^C D_t^{\alpha} B(t)) = 0$ with Caputo fractional derivative.

3 QUANTUM DYNAMICS WITH POWER-LAW MEMORY

Let us consider a generalization of Lindblad equation for quantum observables in the form

$$ {}_0^C D_t^\alpha A_t = -\mathcal{L}_V A_t, \tag{19} $$

where ${}_0^C D_t^\alpha$ is the Caputo fractional derivative with respect to time t (dimensionless variable), and

$$ \mathcal{L}_V A_t = \frac{1}{i\hbar}[H, A_t] - \frac{1}{2\hbar} \sum_{k=1}^{\infty} \Big(V_k^* [A_t, V_k] + [V_k^*, A_t] V_k \Big). \tag{20} $$

For $\alpha = 1$ we have the usual Lindblad equation. If α is non-integer, then equation (19) defines the quantum processes with power-law memory.

If all operators V_k are equal to zero ($V_k = 0$), then we have a generalization of the Heisenberg equation for Hamiltonian system with memory

$$ {}_0^C D_t^\alpha A_t = -\frac{1}{i\hbar}[H, A_t]. \tag{21} $$

Note that the form of $\mathcal{L}_V$ is not uniquely defined. The transformations

$$ V_k \to V_k + a_k I, \quad H \to H + \frac{1}{2i\hbar} \sum_{k=1}^{\infty} (a_k^* V_k - a_k V_k^*), $$

where a_k are arbitrary complex numbers, preserve the form of equation (19).

Cauchy-type problem

If we consider the Cauchy-type problem for equation (19) in which the initial condition is given at the time $t = 0$ by A_0, then its solution can be represented in the form

$$A_t = \Phi_t(\alpha) A_0, \quad (t \geq 0),$$

where

$$\Phi_t(\alpha) = E_\alpha[-t^\alpha \mathcal{L}_V]. \tag{22}$$

Here $E_\alpha[\mathcal{L}]$ is the Mittag-Leffler function with the superoperator argument

$$E_\alpha[\mathcal{L}] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} \mathcal{L}^k.$$

Note that the relation

$${}_a^C D_t^\alpha \, E_\alpha[\lambda(t-a)^\alpha] = \lambda \, E_\alpha[\lambda(t-a)^\alpha]$$

holds for $\lambda \in \mathbb{C}$, $t > a$, $a \in \mathbb{R}$, and $\alpha > 0$.

Quantum dynamical groupoid

The superoperators $\Phi_t(\alpha)$, $t \geqslant 0$, describe dynamics of open quantum systems with power-law memory. The superoperator $\mathcal{L}_V$ can be considered as a generator of the quantum dynamical groupoid $\Phi_t(\alpha)$:

$$ {}_0^C D_t^{\alpha} \Phi_t(\alpha) = -\mathcal{L}_V \Phi_t(\alpha). $$

For $\alpha = 1$, we have $\Phi_t(1) = E_1[-t\mathcal{L}_V] = \exp\{-t\mathcal{L}_V\}$. The superoperators $\Phi_t = \Phi_t(1)$ form a semigroup such that $\Phi_t \Phi_s = \Phi_{t+s}, \quad (t, s > 0), \quad \Phi_0 = L_I$. This property holds since $\exp\{-t\mathcal{L}_V\} \ \exp\{-s\mathcal{L}_V\} = \exp\{-(t+s)\mathcal{L}_V\}$.

For $\alpha \notin \mathbb{N}$ we have

$$ E_\alpha[-t^\alpha \mathcal{L}_V] \ E_\alpha[-s^\alpha \mathcal{L}_V] \neq E_\alpha[-(t+s)^\alpha \mathcal{L}_V]. $$

Therefore the semigroup property is not satisfied for non-integer values of α:

$$ \Phi_t(\alpha)\Phi_s(\alpha) \neq \Phi_{t+s}(\alpha), \quad (t, s > 0). $$

As a result, the superoperators $\Phi_t(\alpha)$ with $\alpha \notin \mathbb{N}$ cannot form a semigroup. This property means that we have a quantum processes with memory. The superoperators $\Phi_t(\alpha)$ describe quantum dynamics of open systems with memory. The memory effects to dynamical maps mean that their present evolution of $A(t)$ depends on all past values of $A(\tau)$ for $\tau < t$.

4 LINEAR OSCILLATOR WITH FRICTION AND MEMORY

Let us consider an oscillator with linear friction and power-like memory. In this example the basic assumption is that the general form of a bounded completely dissipative superoperator holds for an unbounded superoperator $\mathcal{L}_V$. We assume that the operators H, and V_k are functions of the operators Q and P such that the obtained model is exactly solvable. Therefore we consider $V_k = V_k(Q, P)$ as the first-degree polynomials in Q and P, and the Hamiltonian $H = H(Q, P)$ as a second degree polynomial in Q and P:

$$H = \frac{1}{2m}P^2 + \frac{m\omega^2}{2}Q^2 + \frac{\mu}{2}(PQ + QP), \tag{23}$$

$$V_k = a_k P + b_k Q,$$

where a_k, and b_k, $k = 1, 2$, are complex numbers. These assumptions mean that the friction force is proportional to the velocity.

Using the definition of $\mathcal{L}_V$ and the canonical commutation relations for operators Q and P, we obtain

$$-\mathcal{L}_V Q = \frac{1}{m} P + \mu Q - \lambda Q,$$

$$-\mathcal{L}_V P = -m\omega^2 Q - \mu P - \lambda P,$$

where $\lambda = Im\Big(a_1 b_1^* + a_1 b_1^*\Big)$. Let us consider generalized Lindblad equation (19) for Q_t and P_t in the form

$$ {}_0^C D_t^\alpha Q_t = -\mathcal{L}_V Q_t, \quad {}_0^C D_t^\alpha P_t = -\mathcal{L}_V P_t, \tag{24}$$

where ${}_0^C D_t^\alpha$ is the Caputo fractional derivative with respect to time t, which is dimensionless variable.

We define the matrices

$$A = \begin{pmatrix} Q \\ P \end{pmatrix}, \quad M = \begin{pmatrix} \mu - \lambda & m^{-1} \\ -m\omega^2 & -\mu - \lambda \end{pmatrix}. \tag{25}$$

Then equations (24) for quantum observables have the matrix representation

$$ {}_0^C D_t^\alpha A_t = M A_t, \tag{26}$$

where $-\mathcal{L}_V A_t = M A_t$.

If we consider the Cauchy problem for equation (26) in which the initial condition is given at the time $t = 0$ by A_0, then its solution can be represented in the form

$$A_t = \Phi_t(\alpha) A_0,$$

where

$$\Phi_t(\alpha) = E_\alpha[t^\alpha M].$$

The Mittag-Leffler function with the matrix argument is defined by

$$E_\alpha[t^\alpha M] = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(\alpha n + 1)} M^n.$$

For $\alpha = 1$, we obtain

$$\Phi_t(1) = \Phi_t = \sum_{n=0}^{\infty} \frac{t^n}{n!} M^n = e^{tM}. \qquad (27)$$

The matrix M can be represented in the form

$$M = N^{-1} F N, \tag{28}$$

where F is a diagonal matrix, and

$$N = \begin{pmatrix} m\omega^2 & \mu+\nu \\ m\omega^2 & \mu-\nu \end{pmatrix}, \tag{29}$$

$$F = \begin{pmatrix} -(\lambda+\nu) & 0 \\ 0 & -(\lambda-\nu) \end{pmatrix}. \tag{30}$$

Here we use the complex parameter ν, such that $\nu^2 = \mu^2 - \omega^2$.

Using (28), the one-parameter superoperators $\Phi_t(\alpha)$ are represented by

$$\Phi_t(\alpha) = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(\alpha n+1)} M^n = N^{-1} \left(\sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(\alpha n+1)} F^n \right) N.$$

As a result, we have

$$\Phi_t(\alpha) = N^{-1} \, E_\alpha[t^\alpha F] \; N. \tag{31}$$

For $\alpha = 1$, we have

$$\Phi_t(1) = N^{-1} e^{tF} N.$$

Substitution of (29) and (30) into (31) gives

$$\Phi_t(\alpha) = \begin{pmatrix} C_\alpha[\lambda,\nu,t] + (\mu/\nu)\, S_\alpha[\lambda,\nu,t] & (1/m\nu)\, S_\alpha[\lambda,\nu,t] \\ -(m\omega^2/\nu)\, S_\alpha[\lambda,\nu,t] & C_\alpha[\lambda,\nu,t] - (\mu/\nu)\, S_\alpha[\lambda,\nu,t] \end{pmatrix},$$

where we use the notations

$$S_\alpha[\lambda,\nu,t] = \frac{1}{2}\Big(E_\alpha[(-\lambda+\nu)t^\alpha] - E_\alpha[(-\lambda-\nu)t^\alpha]\Big),$$

$$C_\alpha[\lambda,\nu,t] = \frac{1}{2}\Big(E_\alpha[(-\lambda+\nu)t^\alpha] + E_\alpha[(-\lambda-\nu)t^\alpha]\Big).$$

As a result, we obtain $A_t(\alpha) = \Phi_t(\alpha)A_0$ **in the form**

$$Q_t = \Big(C_\alpha[\lambda,\nu,t] + \frac{\mu}{\nu}\, S_\alpha[\lambda,\nu,t]\Big)Q_0 + \frac{1}{m\nu}\, S_\alpha[\lambda,\nu,t]P_0, \qquad (32)$$

$$P_t = -\frac{m\omega^2}{\nu}\, S_\alpha[\lambda,\nu,t]Q_0 + \Big(C_\alpha[\lambda,\nu,t] - \frac{\mu}{\nu}\, S_\alpha[\lambda,\nu,t]\Big)P_0. \qquad (33)$$

For $\alpha = 1$**, we get**

$$S_\alpha[\lambda,\nu,t] = e^{-\lambda t}\sinh(\nu t), \quad C_\alpha[\lambda,\nu,t] = e^{-\lambda t}\cosh(\nu t),$$

and equations (32) and (33) give the well-known solutions.

Note that for $V_k = 0$ $(k = 1, 2)$, we have closed Hamiltonian system with memory, that is described by Heisenberg equation (21). The solution of the equation for linear oscillator with power-law memory is given by (32) and (33), where $\lambda = 0$.

For non-integer α, the Mittag-Leffler function in equations (32) and (33) can be represented in the form

$$E_{\alpha,1}(-zt^{\alpha}) = f_{\alpha}(z^{1/\alpha}t) + g_{\alpha}(z^{1/\alpha}t), \tag{34}$$

where

$$f_{\alpha}(t) = \frac{1}{\pi}\int_0^{\infty} e^{-rt}\frac{r^{\alpha-1}\sin(\pi\alpha)}{r^{2\alpha}+2r^{\alpha}\cos(\pi\alpha)+1}dr,$$

$$g_{\alpha}(t) = \frac{2}{\alpha}e^{t\cos(\pi/\alpha)}\ \cos\left[t\sin(\pi/\alpha)\right]. \tag{35}$$

The function $g_{\alpha}(t)$ exhibits oscillations with circular frequency $\Omega(\alpha) = \sin(\pi/\alpha)$, and exponentially decaying amplitude with rate $\lambda(\alpha) = |\cos(\pi/\alpha)|$. The functions $f_{\alpha}(t)$ exhibit an algebraic decay as $t \to \infty$. Therefore the linear oscillator with memory demonstrates power-law decay. Note that we have power-law decay for open and closed Hamiltonian quantum systems with memory. As a result, the power-law memory leads to dissipation.

As a result, generalized Lindblad equation with time fractional derivative describes evolution of quantum observables of open quantum systems with memory. The quantum processes with power-law memory ($\alpha \notin \mathbb{N}$) cannot be described by a semigroup. It can be described only as a quantum dynamical groupoid. As a result, the long-term memory for open and closed quantum systems can lead to dissipation with power-law decay.

V.E. Tarasov, *Quantum Mechanics of Non-Hamiltonian and Dissipative Systems* **(Elsevier, Amsterdam, London, 2008)**

V.E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media* **(Springer, New York, 2011)**